Markovian Queues in Random Environment
with System Failures

Noam Paz
Advisor: Prof. Uri Yechiali

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Department of Statistics and Operations Research
School of Mathematical Sciences
Tel-Aviv University

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I would like to dedicate this work to my wife Giselle, who has always been there for me, and to our princess Ayala who came to this world around the same time as this thesis.
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1 Abstract

We study two Markovian queueing systems in random environment: an M/M/∞ queue, and an M/M/1 queue with system failures. The underlying random environment is an $n$-phase continuous-time Markov chain, and in both models, when in phase $i$, the arrival and service rates are $\lambda_i$ and $\mu_i$, respectively, while the residence time is exponentially distributed with parameter $\eta_i$.

For the $M/M/\infty$ queue we calculate the stationary probabilities of the system residing in phase $i$, $1 \leq i \leq n$, derive the (conditional) probability generating functions (PGFs) of the system when in phase $i$, and calculate mean queue sizes. For the special case when $\lambda_i/\mu_i = c$ for all $i$, we derive explicit simple solution of the steady state probabilities and show that the system as a whole behaves like a mashed single-phase $M/M/\infty$ queue if and only if $\lambda_i/\mu_i = c$ for all $i$. For the general case we present numerical results and graphs demonstrating the relations between the given parameters.

For the $M/M/1$ queue in random environment we consider the case of random disastrous system’s failures (catastrophes) that, when occur, cause all the customers present to be cleared out of the system and the system itself to move to a repair phase, $i = 0$, whose duration is exponentially distributed with parameter $\eta_0$. As soon as the system is repaired, it moves to phase $i \geq 1$ with probability $q_i$, $\sum_{i=1}^{n} q_i = 1$. We derive the conditional PGFs of the various phases, calculate mean queue sizes, and derive the mean sojourn time of an arbitrary customer. Two special cases are further discussed: (i) when arrivals stop during the period in which the system is being repaired, and (ii) when the system is comprised of only two phases having identical arrival rates. Numerical results and graphs are presented.
2 Introduction

The traffic in roadways is influenced by many factors. In recent years vehicles have become safer while having more power, and the roads have become wider. These and other factors affect the road traffic, the congestion of vehicles on the road and their speed. Vehicles are much faster, and this may cause a road section to contain more vehicles every second. Therefore, every incident on the roadway may cause a larger number of vehicles to be affected: they have to decrease their speed, or they may not be able to pass the road section at all.

Road traffic statistics can be expressed in terms of queueing models, assuming the arrival rate of vehicles is a Poisson process, and interpreting the movement on a road section as a service. The various queueing models differ by their arrival rate, regime of the queue, number of servers, service rate, etc. In this work we analyze two models of queueing systems in random environment which approximately represent vehicular traffic (see, for example, Baykal-Gursoy, Xiao and Ozbay [2]).

Our first model studies a queueing system in random environment that includes $n$ phases with infinite number of servers in each phase. Every phase is a $M/M/\infty$ queue with different arrival rate and service intensity. O’Cinneide and Purdue [12], and Keilson and Servi [23] analyzed the $n$-state $M/M/\infty$ queue. However, no explicit solutions were given. For the special case of $M/M/\infty$ queue with two-state Markov modulated arrival process, Keilson and Servi [8] show that the decomposition property holds, and provided the explicit solution. Baykal-Gursoy and Xiao [1] also discussed a 2-phase case of the system, in which the service rate is smaller in a case of failure, but service still continues. They dealt with system breakdowns, as well as partial failures, where the servers work at a lower efficiency and derived explicit results for the steady-state probabilities. Baykal-Gursoy, Xiao and Ozbay [2] analyzed a $M/M/c$ queueing system under partial and complete service interruptions, and gave simulated results showing that their model is applicable for $M/M/\infty$ queues for large values of $c$. D’Auria [4] studied a two-phase system of $M/M/\infty$ queue, and discussed a special case, when there is no service upon failure. Yechiali [15] considered a typical $M/M/1$ queueing model in a random environment with $n$ phases. Each phase uses different arrival
rate and service intensity. The focus in [15] is on the steady-state regime, and special cases are analyzed. Mitrany and Avi-Itzhak [10] analyzed a system of multi-server queue, in which every server may be down independently of the others. They discussed the $M/M/c$ queue and obtained the generating function of the queue size for $c \leq 2$. They also suggested a numerical algorithm for calculations when $c > 2$. Neuts [11] presented the $M/M/\infty$ system in random environment as a matrix-geometric queue. Our model is, again, a $M/M/\infty$ queue in random environment but in spite of Neuts’s ’complaint’ that this model is surprisingly resistent to analytic solution, we are able to calculate explicitly the conditional mean queue sizes of the system when in a given environment. Furthermore, we show that the system as a whole behaves like a single-phase $M/M/\infty$ queue if and only if, for each environment, the ratio of arrival rate to service rate is the same. That is, what matters is the above ratio and not the actual values of the arrival and service rate in each phase. We also analyze various extreme cases by changing the time durations that the system resides in the various phases.

As indicated, the motivation for studying this system comes from road traffic area. Consider a road section with incidents (’failures’) that slow down the traffic. The service time of a customer is equivalent to the duration of time it takes for a vehicle to drive along a road section. Thus, in a road of several kilometers and number of lanes, there are hundreds and thousands of vehicles, each one travels at its own speed, so that the process may be approximated and modeled as a $M/M/\infty$ queueing system. In a case of an incident in a given lane of the road, all vehicles slow their speed, until the incident is cleared. An ’incident’ may be caused by an accident, traffic light problems, a broken car which blocks a lane, etc.

Another model we deal with is a $M/M/1$ queue system in random environment and with failures. The failures cause a ”clear and lost” situation: when there is an incident, all customers are lost and there is no service until the incident is fixed. Cooper and Murray [3] analyzed a queueing system with a single server that serves two queues by a circular order, with a general service distribution and Poissonian arrival time. Levy and Yechiali [9], Fuhrmann [5] and Fuhrmann and Cooper [6] studied $M/G/1$ queues with different cases of server vacations. They showed that the stationary distribution of the number of customers in a $M/G/1$ queueing system with server vacations is the convolution of the distribution
functions of two independent random variables (decomposition property). Yadin and Naor [14] studied a service model with two queue phases, when one of the arrival rates, \( \mu_1 \), may be equal to zero. Harris and Marchal [7] generalized the \( M/G/1 \) model with server vacations and found necessary and sufficient conditions for an ergodic system. In our model we study a case of 'clear and lost' customers. We also analyze a special case where there is no arrival of customers at the failure phase. We calculate the mean queue sizes and probabilities of staying in phase \( i \).

The motivation for this second model is also taken from the transportation area. In case of an incident, such as an accident or a broken vehicle, the vehicles are evacuated to another roadway (in our model’s words, they are lost), and new vehicles that arrive to the road wait until the road is fixed, and ready again for traveling.

The structure of the work is as follows: after section 2 (Introduction) we deal in section 3 with the \( M/M/\infty \) queueing system. We formulate the model, derive its balance equations, and obtain stability conditions for the system. We then calculate the mean queue sizes using generating functions. We also study special cases such as a fixed proportion of arrival intensity to service rate, \( \lambda_i/\mu_i \), and extreme cases of transition rates between phases. We show that the system as a whole behaves like a single \( M/M/\infty \) queue if and only if \( \lambda_i/\mu_i \) is equal for all phases. A few numerical results are presented. Section 4 deals with the \( M/M/1 \) queueing system with system failures. First we formulate the problem and describe the model. Then we find the stability conditions, and calculate mean queue sizes, using generating functions. We further study a special case in which there are no arrivals when the system is down. Numerical results are presented. In section 5 we give a transportation implementation to the models. Graphs showing the dependence of mean queue size on the vehicles’ speed are presented. In an Appendix we further discuss the issue of driving speeds and their influence on the throughput of a roadway.
3 An infinite-server queueing system
in random environment

3.1 The model and balance equations

Consider an M/M/$\infty$ type queue operating in 'random environment' for which the underlying process is an $n$-dimensional continuous-time Markov chain (MC). That is, when the process is in phase $i$, the system functions as an $M(\lambda_i)/M(\mu_i)/\infty$ queue, with Poisson arrival rate $\lambda_i$ and service rate $\mu_i$ by each server.

The duration of time the MC (and the system) stays in phase $i$ is an exponentially distributed random variable with mean $1/\eta_i$. When the system ends its sojourn period in phase $i$, it jumps (instantaneously) to phase $j$ with probability $q_{ij}$ ($i, j = 1, 2, \ldots, n$), $\sum_{j=1}^{n} q_{ij} = 1$ $\forall i$. We denote the phase-transition matrix of the underlying MC by $Q \equiv [q_{ij}]$, and assume w.l.o.g. that $q_{ii} = 0$ $\forall i$.

A stochastic process $\{U(t), X(t)\}$ describes the system’s state at time $t$ as follows: $U(t)$ denotes the phase in which the system operates at time $t$, while $X(t)$ counts the number of customers present in the system at that time. The system is said to be in state $(i, m)$ if it is in phase $i$, and there are $m$ customers in the system. Accordingly, let $p_{im}$ be the steady-state probability of the system in state $(i, m)$. That is, $p_{im} = P(U(t) = i, X(t) = m)$ $\forall t \geq 0$, $1 \leq i \leq n$, $m = 0, 1, 2, \ldots$.

Figure 1 below depicts a transition-rate diagram of the described queueing system.

The steady-state balance equations are given as follows:

For $i = 1, 2, \ldots, n$ and $m = 0$,

$$(\lambda_i + \sum_{j=1}^{n} \eta_i q_{ij})p_{i0} = \mu_i p_{i1} + \sum_{j=1}^{n} \eta_j q_{ji} p_{j0}$$

(3.1)

For $i = 1, 2, \ldots, n; m \geq 1$,

$$(\lambda_i + m \mu_i + \sum_{j=1}^{n} \eta_i q_{ij})p_{im} = \lambda_i p_{i,m-1} + (m + 1) \mu_i p_{i,m+1} + \sum_{j=1}^{n} \eta_j q_{ji} p_{jm}$$

(3.2)
\[X(t) = \begin{array}{cccc}
0 & 1 & 2 & \ldots & m-1 & m
\end{array}\]

\[U(t) = 1\]

\[\begin{array}{cccc}
\lambda_1 & \lambda_1 & \lambda_1 & \ldots & \lambda_1 & \lambda_1 \\
\mu_1 & 2\mu_1 & \lambda_2 & \ldots & \lambda_n & \lambda_n \\
\mu_2 & 2\mu_2 & \mu_1 & \ldots & \mu_n & \mu_n \\
\mu_3 & \mu_3 & \mu_3 & \ldots & \mu_3 & \mu_3 \\
\mu_4 & \mu_4 & \mu_4 & \ldots & \mu_4 & \mu_4 \\
\mu_n & \mu_n & \mu_n & \ldots & \mu_n & \mu_n \\
\end{array}\]

Figure 1: A transition-rate diagram of an infinite-server queueing system in random environment. Transitions between phases are shown only for \(X(t) = 0\) and \(X(t) = m - 1\).

Define \(\eta_{ij} = \eta_{q_{ij}}\) and \(\lambda_i = 0\) for \(m < 0\). Then equations (3.1) and (3.2) can be written as

\[(\lambda_i + m\mu_i + \eta_i)p_{im} = \lambda_ip_{i,m-1} + (m + 1)\mu_ip_{i,m+1} + \sum_{j=1}^{n} \eta_{ji}p_{jm} \quad \forall i = 1, 2, \ldots, n; \quad m \geq 0 \quad (3.3)\]

Summing equation (3.3) over all \(m\) gives

\[\eta_{i}p_{i\ast} = \sum_{j=1}^{n} \eta_{ji}p_{j\ast\ast} \quad (3.4)\]

where \(p_{i\ast} = \sum_{m=0}^{\infty} p_{im}\).

The limit probabilities of the underlying Markov chain \(Q\), describing just the random environment process, are \(\pi_j = P(U(t) = j)\). The \(\pi_j\)'s satisfy \(\sum_{j=1}^{n} \pi_j = 1\) and \(\pi_j = \sum_{i=1}^{n} \pi_iq_{ij}\),
and are independent of the values of \( \{\lambda_i\} \), \( \{\mu_i\} \) and \( \{\eta_i\} \). The proportion of time the system resides in phase \( i \) is given by

\[
p_i = \frac{\pi_i}{\sum_{k=1}^{n} \eta_k \pi_k}
\]

(3.5)

Evidently, the \( \{p_i\} \) are independent of \( \{\lambda_i\} \) and \( \{\mu_i\} \).

### 3.2 Stability conditions

Summing equation (3.3) over all \( i \) and canceling terms gives

\[
\sum_{i=1}^{n} \lambda_i p_{im} = \sum_{i=1}^{n} (m + 1) \mu_i p_{i,m+1} \quad m = 0, 1, 2, \ldots
\]

(3.6)

Summing equation (3.6) over all \( m \) yields

\[
\sum_{m=0}^{\infty} \sum_{i=1}^{n} \lambda_i p_{im} = \sum_{m=1}^{\infty} \sum_{i=1}^{n} m \mu_i p_{im}
\]

(3.7)

After defining

\[
\hat{\mu} = \sum_{i=1}^{n} \sum_{m=0}^{\infty} m \mu_i p_{im} = \sum_{i=1}^{n} \mu_i \sum_{m=0}^{\infty} m p_{im} = \sum_{i=1}^{n} \mu_i E[L_i]
\]

where \( E[L_i] = \sum_{m=0}^{\infty} m p_{im} \), and

\[
\hat{\lambda} = \sum_{i=1}^{n} \sum_{m=0}^{\infty} \lambda_i p_{im} = \sum_{i=1}^{n} \lambda_i p_i
\]

we get, using equation (3.7),

\[
\hat{\mu} - \hat{\lambda} = \sum_{i=1}^{n} \sum_{m=0}^{\infty} m \mu_i p_{im} - \sum_{i=1}^{n} \sum_{m=0}^{\infty} \lambda_i p_{im}
\]

\[
= \sum_{m=1}^{\infty} \sum_{i=1}^{n} m \mu_i p_{im} - \sum_{i=1}^{n} \sum_{m=0}^{\infty} \lambda_i p_{im}
\]

\[
= \sum_{m=0}^{\infty} \sum_{i=1}^{n} \lambda_i p_{im} - \sum_{i=1}^{n} \sum_{m=0}^{\infty} \lambda_i p_{im} = 0
\]

(3.8)

That is, in contrast with the \( M/M/1 \) queue in random environment, where stability holds if and only of \( \hat{\mu} \equiv \sum_{i=1}^{n} \mu_i p_i > \hat{\lambda} \) (see e.g. Yechiali [15]), in the \( M/M/\infty \) queue, the system is always stable.
3.3 Generating functions and mean queue sizes

We now use (partial) generating functions to find the unknown set of probabilities \( \{p_{im}\} \), and in particular the probabilities \( \{p_{i0}\} \).

Let

\[
G_i(z) = \sum_{m=0}^{\infty} z^m p_{im} \quad i = 1, 2, \ldots, n
\]  

be the (partial) generating functions of phase \( i \). Multiplying both sides of equation (3.3) by \( z^m \) and summing over all \( m \) yield a system of \( n \) differential equations in the \( n \) unknowns \( G_i(z) \):

\[
\mu_i (1 - z) G_i'(z) = (\lambda_i (1 - z) + \eta_i) G_i(z) - \sum_{j=1}^{n} \eta_{ji} G_j(z) \quad i = 1, 2, \ldots, n
\]  

Equation (3.10) can be written as a matrix differential equation:

\[
A(z)G'(z) = B(z)G(z)
\]

where the matrices \( A(z) \) and \( B(z) \) are given by

\[
A(z) = \begin{pmatrix}
\mu_1 (1 - z) & 0 & 0 & \ldots & 0 \\
0 & \mu_2 (1 - z) & 0 & \ldots & 0 \\
& \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ldots & \mu_n (1 - z)
\end{pmatrix}
\]

and

\[
B(z) = \begin{pmatrix}
\lambda_1 (1 - z) + \eta_1 & -\eta_{21} & -\eta_{31} & \ldots & -\eta_{n1} \\
-\eta_{12} & \lambda_2 (1 - z) + \eta_2 & -\eta_{32} & \ldots & -\eta_{n2} \\
& \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
-\eta_{1n} & -\eta_{2n} & -\eta_{3n} & \ldots & \lambda_n (1 - z) + \eta_n
\end{pmatrix}
\]

and \( G(z) \) is a \( n \)-dimensional vector: \( G(z) = (G_1(z), G_2(z), \ldots, G_n(z))^T \).
Note that $A(z)$ is singular at $z = 1$. However, for $0 \leq z < 1$, the above can be written as

\[ G'(z) = C(z)G(z) \]

where

\[ C(z) = A^{-1}(z)B(z) \]

\[
\begin{pmatrix}
\lambda_1(1 - z) + \eta_1 & -\eta_2 & -\eta_3 & \cdots & -\eta_n \\
\mu_1(1 - z) & \lambda_2(1 - z) + \eta_2 & \mu_1(1 - z) & \cdots & \mu_1(1 - z) \\
-\eta_{12} & \mu_2(1 - z) & \lambda_2(1 - z) + \eta_2 & \cdots & \mu_1(1 - z) \\
\mu_2(1 - z) & \mu_2(1 - z) & \mu_2(1 - z) & \cdots & \mu_2(1 - z) \\
& \ddots & \ddots & \ddots & \ddots \\
& & -\eta_{1n} & -\eta_{2n} & -\eta_{3n} & \ldots & \lambda_n(1 - z) + \eta_n \\
& & \mu_n(1 - z) & \mu_n(1 - z) & \mu_n(1 - z) & \cdots & \mu_n(1 - z)
\end{pmatrix}
\]

(3.11)

Apparently, there is no simple analytic solution to the above set. Indeed, Neuts ([11], page 274) states the following: "We note that the infinite-server queue $M/M/\infty$ in a Markovian environment is surprisingly resistant to analytic solution... Brute force numerical solution of a truncated version of the birth-and-death equations enables one to solve this model for a wide range of parameter values in spite of the lack of a mathematically elegant solution."

However, one can calculate mean queue sizes as follows:

Differentiating equation (3.10) yields

\[ -\mu_i G_i'(z) + \mu_i (1 - z) G_i''(z) = -\lambda_i G_i(z) + (\lambda_i (1 - z) + \eta_i) G_i'(z) - \sum_{j=1}^{n} \eta_{ji} G_j'(z) \quad i = 1, 2, \ldots, n \]

(3.12)

By setting $z = 1$ we get

\[ -\mu_i E[L_i] = -\lambda_i G_i(1) + \eta_i E[L_i] - \sum_{j=1}^{n} \eta_{ji} E[L_j] \]

(3.13)

where $G_i'(1) = E[L_i] = \sum_{m=0}^{\infty} mp_{im}$.

Also, $G_i(1) = \sum_{m=0}^{\infty} p_{im} = p_i$. Then, from equation (3.13),

\[ (\eta_k + \mu_i) E[L_i] - \sum_{j=1; j \neq i}^{n} \eta_{ji} E[L_j] = \lambda_i p_i \quad i = 1, 2, \ldots, n \]

(3.14)
This equation can be written as a matrix equation

\[ DE[L] = b \]  \hspace{1cm} (3.15)

where the matrix \( D \) is given by

\[
D = \begin{pmatrix}
\eta_1 + \mu_1 & -\eta_{21} & -\eta_{31} & \cdots & -\eta_{n1} \\
-\eta_{12} & \eta_2 + \mu_2 & -\eta_{32} & \cdots & -\eta_{n2} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-\eta_{1n} & -\eta_{2n} & -\eta_{3n} & \cdots & \eta_n + \mu_n
\end{pmatrix}
\]

and the vectors \( E[L] \) and \( b \) are given by

\[
E[L] = \begin{pmatrix}
E[L_1] \\
E[L_2] \\
\vdots \\
E[L_n]
\end{pmatrix}, \quad \quad b = \begin{pmatrix}
\lambda_1 p_1 \\
\lambda_2 p_2 \\
\vdots \\
\lambda_n p_n
\end{pmatrix}
\]

Therefore the solution of the system is given by

\[ E[L] = D^{-1}b \]  \hspace{1cm} (3.16)

The expected value of the total number of customers in the system is \( E[L] = \sum_{i=1}^{n} E[L_i] \), and the mean sojourn time of an arbitrary customer is, by Little’s law, \( E[W] = \frac{1}{\hat{\lambda}} E[L] \), where \( \hat{\lambda} = \sum_{i=1}^{n} \lambda_i p_i \).

Examples: a. When \( n = 1 \), i.e. when the system shrinks to a single phase, then \( D = \Delta(\eta + \mu) = (\eta_1 + \mu_1) \), where \( \Delta(a) \) is the diagonal matrix of \( a \), for \( a = (a_1, a_2, \ldots, a_n)^T \). For the single phase we have \( \eta_{11} = \eta_1 = 0 \) and \( p_1 = 1 \). Using (3.15) we get

\[ (\eta_1 + \mu_1) E[L_1] = \lambda_1 p_1 \]  \hspace{1cm} (3.17)
which leads to

\[ E[L_1] = \frac{\lambda_1}{\mu_1} \quad (3.18) \]

Indeed, for an \( M(\lambda_1)/M(\mu_1)/\infty \) system, the mean queue size is \( E[L_1] = \frac{\lambda_1}{\mu_1} \).

b. When \( n = 2 \), equation (3.15) leads to

\[
\begin{pmatrix}
\eta_1 + \mu_1 & -\eta_{21} \\
-\eta_{12} & \eta_2 + \mu_2
\end{pmatrix}
\begin{pmatrix}
E[L_1] \\
E[L_2]
\end{pmatrix}
= \begin{pmatrix}
\lambda_1 p_1, \\
\lambda_2 p_2,
\end{pmatrix}
\]

(3.19)

Multiplying both sides by the inverse matrix of \( D \) we get

\[
\begin{pmatrix}
E[L_1] \\
E[L_2]
\end{pmatrix}
= \frac{1}{(\eta_1 + \mu_1)(\eta_2 + \mu_2) - \eta_{12}\eta_{21}}
\begin{pmatrix}
\eta_2 + \mu_2 & \eta_{21} \\
\eta_{12} & \eta_1 + \mu_1
\end{pmatrix}
\begin{pmatrix}
\lambda_1 p_1, \\
\lambda_2 p_2,
\end{pmatrix}
\]

(3.20)

Hence, for \( n = 2 \),

\[
\begin{pmatrix}
E[L_1] \\
E[L_2]
\end{pmatrix}
= \frac{1}{(\eta_1 + \mu_1)(\eta_2 + \mu_2) - \eta_{12}\eta_{21}}
\begin{pmatrix}
\lambda_1 p_1, (\eta_2 + \mu_2) + \lambda_2 p_2, \eta_{21} \\
\lambda_2 p_2, (\eta_1 + \mu_1) + \lambda_1 p_1, \eta_{12}
\end{pmatrix}
\]

(3.21)

3.4 Numerical results

We present a few numerical examples of the model introduced in this section. We set the number of phases to be \( n = 4 \) and calculate \( \{\pi_i\}, \{p_i\} \) and \( \{E[L_i]\} \) for various values of the parameters.

(i) \( \lambda_i = 0.4 \) for all \( i \)

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<th>( \mu_i )</th>
<th>( \eta_i )</th>
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\[ E[L] = 0.7796 \]
(ii) $\mu_i = 0.5$ for all $i$

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<th>$\mu_i$</th>
<th>$\eta_i$</th>
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$E[L] = 1.1519$

(iii) $\eta_i = 0.05$ for all $i$. Clearly, in this case $\pi_i = p_i$.

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$E[L] = 1.8311$

3.5 A special case: $\lambda_i/\mu_i = c$ for all $i$

An interesting special case is when the ratios between the arrival rate and the service rate, $\frac{\lambda_i}{\mu_i}$, are the same for all phases. We will show that if $\frac{\lambda_i}{\mu_i} = c$ for every phase $i$, then the system possesses properties of a standard $M/M/\infty$ queue, and an explicit simple solution can be derived. We state the following:

**Theorem 1**

$$p_{im} = p_i p_m = p_i e^{-c \frac{m}{m!}}$$

if and only if, for every $i = 1, 2, \ldots, n$,

$$\frac{\lambda_i}{\mu_i} = c$$

(3.22)
**Proof:** The proof will be carried via a sequence of lemmas leading to result (3.22).

**Lemma 2** If
\[
\frac{\lambda_i}{\mu_i} = c
\]
then
\[
p_{im} = p_i e^{-c m} \frac{e^m}{m!} \quad i = 1, \ldots, n; \quad m = 0, 1, 2, \ldots
given by (3.23)
\]

**Proof:** Assume \( \frac{\lambda_i}{\mu_i} = c \) \( \forall i \). Adding same terms to both sides of (3.4) we write
\[
(\lambda_i + m\mu_i + \eta_i)p_i = m\mu_ip_i + \lambda_ip_i + \sum_{j=1}^{n} \eta_ip_j.
\]
Multiplying by \( c^n \) and using the assumption \( \frac{\lambda_i}{\mu_i} = c \) yields
\[
(\lambda_i + m\mu_i + \eta_i)c^n = m\lambda_ip_i c^{n-1} + \mu_ip_i c^{n+1} + \sum_{j=1}^{n} \eta_ip_j c^n
\]
Dividing by \( m! \) and multiplying by \( e^{-c} \) leads to
\[
(\lambda_i + m\mu_i + \eta_i)e^{-c} c^m \frac{c^m}{m!} = \lambda_ip_i c^{m-1} e^{-c} \frac{c^{m-1}}{(m-1)!} + (m+1)\mu_ip_i c^{m+1} e^{-c} \frac{c^{m+1}}{(m+1)!} + \sum_{j=1}^{n} \eta_ip_j e^{-c} \frac{c^m}{m!}
\]
Setting \( p_{im} = p_i e^{-c} \frac{c^m}{m!} \) in (3.26) leads to the steady-state balance equation (3.3). Since equations (3.3) and (3.4) possess a unique solution, \( p_{im} = p_i e^{-c} \frac{c^m}{m!} \) is the one.

**Lemma 3** If
\[
\frac{\lambda_i}{\mu_i} = c
\]
then
\[
p_{im} = p_im
\]

**Proof:** Assume
\[
\frac{\lambda_i}{\mu_i} = c
\]
Using lemma 2 and summing equation (3.23) for all \(i\) leads to

\[
p_{m} = e^{-c} \frac{c^m}{m!} \quad \forall m
\]

(3.27)

Substituting (3.27) in (3.23) completes the proof.

\[\rule{0.5cm}{0.1cm}\]

**Lemma 4** If

\[
p_{im} = p_{i}p_{m} \quad i = 1, \ldots, n; \quad \forall m
\]

then

\[
p_{m} = e^{-\frac{\lambda_i}{\mu_i} \left( \frac{\lambda_i}{\mu_i} \right)^m}{m!} \quad i = 1, \ldots, n; \quad \forall m
\]

and

\[
\frac{\lambda_i}{\mu_i} = c \quad i = 1, \ldots, n
\]

**Proof:** Substituting \(p_{im} = p_{i}p_{m}\) in equation (3.3) gives

\[
(\lambda_i + m\mu_i + \eta_i)p_{i}p_{m} = \lambda_i p_{i}p_{m-1} + (m + 1)\mu_i p_{i}p_{m+1} + \sum_{j=1}^{n} \eta_{ij} p_{j}p_{m}
\]

(3.28)

By using equation (3.5) and the definition \(\eta_{ij} = \eta_{ij}\) we get

\[
(\lambda_i + m\mu_i + \eta_i)\frac{\pi_i}{\eta_i} p_{m} = \lambda_i \frac{\pi_i}{\eta_i} p_{m-1} + (m + 1)\mu_i \frac{\pi_i}{\eta_i} p_{m+1} + \sum_{j=1}^{n} \eta_{ij} q_{ij} \frac{\pi_j}{\eta_j} p_{m}
\]

(3.29)

Multiplying both sides of (3.29) by \(\eta_i\) and using \(\pi_j = \sum_{i=1}^{n} \pi_{ij}\) yields

\[
(\lambda_i + m\mu_i + \eta_i)\pi_i p_{m} = \lambda_i \pi_i p_{m-1} + (m + 1)\mu_i \pi_i p_{m+1} + \eta_i \pi_i p_{m}
\]

(3.30)

Dividing by \(\pi_i\) leads to

\[
(\lambda_i + m\mu_i) p_{m} = \lambda_i p_{m-1} + (m + 1)\mu_i p_{m+1}
\]

(3.31)

Equation (3.31) is the steady-state balance equation of a standard \(M/M/\infty\) queue. That is, the marginal distribution of \(L\), given \(U(t) = i\), is Poissonian, namely,

\[
p_{m} = e^{-\frac{\lambda_i}{\mu_i} \left( \frac{\lambda_i}{\mu_i} \right)^m}{m!}
\]

(3.32)
Since $p_m$ is independent of the phase $i$, we must have that $\frac{\lambda_i}{\mu_i} = c$ for all $i$. This completes the proof.

Lemmas 3 and 4 now complete the proof of theorem 1.

Corollary 5 If $\frac{\lambda_i}{\mu_i} = c$, $i = 1, \ldots, n$ then the mean number of jobs in the system is given by $E[L] = c$

Proof: Assume $\frac{\lambda_i}{\mu_i} = c$, $i = 1, \ldots, n$. By lemma 3

$$p_m = e^{-\frac{cm}{m!}} \forall m$$

Thus,

$$E[L] = \sum_{m=0}^{\infty} mp_m = \sum_{m=0}^{\infty} m e^{-\frac{cm}{m!}} = c$$

That is, $E[L]$, the mean total number of customers in the system increases linearly with $c$. Furthermore, when $\frac{\lambda_i}{\mu_i} = c$ for all $i$, each $E[L_i]$ also increases linearly with $c$. To see that it is enough to write $b$ in equation (3.15) as $b = c(\mu_1 p_1, \mu_2 p_2, \ldots, \mu_n p_n)^T$ so that $E[L] = cD^{-1}(\mu_1 p_1, \mu_2 p_2, \ldots, \mu_n p_n)^T$.

3.6 Extreme cases of $\eta_i$

We now investigate a few extreme cases relating to the $\eta_i$’s.

a. Consider the case where, for some $i$, $\eta_i \rightarrow 0$, but $\eta_j > 0 \forall j \neq i$. Then using equation (3.5)

$$p_i = \frac{\pi_i}{\eta_i + \sum_{k=1,k\neq i}^{n} \frac{\pi_k}{\eta_k}} = \frac{\pi_i}{\eta_i + \sum_{k=1,k\neq i}^{n} \frac{\pi_k}{\eta_k}} \rightarrow 1$$
Similarly, \( p_j \xrightarrow{\eta_i \to 0} 0 \) for every \( j \neq i \).

Indeed, when \( \eta_i \to 0 \) the system will (almost) always stay in phase \( i \), and the proportion of time it will stay in another phase tends to 0.

b. Suppose \( \eta_i \to \infty \), while \( \eta_j > 0 \ \forall j \neq i \). Again, using equation (3.5), we have

\[
p_i^* = \frac{\pi_i}{\eta_i} \xrightarrow{\eta_i \to 0} 0, \quad p_j^* = \frac{\pi_j}{\eta_j} \xrightarrow{\eta_i \to \infty} 0
\]

That is, the proportion of time the system stays in phase \( i \) tends to 0 and the system behaves as if it consists of only \((n - 1)\) phases.
4 A single server queue in random environment with system failures

4.1 The model and balance equations

Consider an M/M/1 type queue operating in a special 'random environment' underlined by an \((n+1)\)-dimensional continuous-time Markov chain, with phases \(i = 0, 1, 2, \ldots, n\), governed by the matrix \(Q\) of transmission probabilities given as

\[
Q = \begin{pmatrix}
0 & q_1 & q_2 & \cdots & q_n \\
1 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]

When in phase \(i \geq 1\) the system acts as an \(M(\lambda_i)/M(\mu_i)/1\) queue, with Poisson arrival rate \(\lambda_i \geq 0\) and service rate \(\mu_i \geq 0\). The duration of time the system resides in phase \(i\) is an exponentially distributed random variable with mean \(1/\eta_i, i = 1, 2, \ldots, n\).

Moreover, when in phase \(i \geq 1\), the system suffers occasionally a disastrous failure causing it to move to a 'failure' phase, denoted by \(i = 0\). A disaster causes all the customers in the system to be lost, such that the system as a whole moves to state \((U(t) = 0, X(t) = 0)\). When in the failure phase, the system undergoes a repair process, having exponentially distributed duration with mean \(1/\eta_0\). The arrival process continues with rate \(\lambda_0 \geq 0\) but no service is rendered, i.e. \(\mu_0 = 0\). When the system is repaired, it jumps from the failure phase to some phase \(i \geq 1\) with probability \(q_i, \sum_{i=1}^{n} q_i = 1\). That is, there are no direct moves from phase \(i \geq 1\) to phase \(j \geq 1\): in each 'active' phase the system stays until a breakdown occurs, which sends it to phase 0. Only then, after a repair duration, the system can move to one of the phases \(i = 1, 2, \ldots, n\).

Figure 2 below depicts the transition-rate diagram of the above queueing system.
\[
X(t) = 0 \quad 1 \quad 2 \quad m - 1 \quad m
\]

\[
U(t) = 0
\]

\[
\eta_0, \eta_i, \eta_n
\]

\[
i
\]

\[
n
\]

\[
\lambda_0, \lambda_i, \lambda_n \quad \mu_i, \mu_n
\]

\[
\text{Figure 3: A transition-rate diagram of a single-server queueing system in random environment and with system failures. In case of a disaster the system moves to phase } U(t) = 0 \text{ and all customers are lost. After the system is repaired, it moves to phase } i \text{ with probability } q_i. \text{ (Note: for clarity of exposition, not all transitions are shown)}
\]

The steady-state balance equations are given as follows:

For the failure phase \( i = 0 \), while \( m = 0 \),

\[
(\lambda_0 + \eta_0) p_{00} = \sum_{i=1}^{n} \eta_i \sum_{m=0}^{\infty} p_{im} = \sum_{i=1}^{n} \eta_i p_i. \quad (4.1)
\]

and while \( m \geq 1 \)

\[
(\lambda_0 + \eta_0) p_{0m} = \lambda_0 p_{0,m-1} \quad (4.2)
\]

For \( i = 1, 2, \ldots, n \) and \( m = 0 \),

\[
(\lambda_i + \eta_i) p_{i0} = \mu_i p_{i1} + \eta_0 q_i p_{00} \quad (4.3)
\]

and when \( m \geq 1 \),

\[
(\lambda_i + \mu_i + \eta_i) p_{im} = \lambda_i p_{i,m-1} + \mu_i p_{i,m+1} + \eta_0 q_i p_{0m} \quad (4.4)
\]
From (4.1) and (4.2) we get that

\[ p_{0m} = \left( \frac{\lambda_0}{\lambda_0 + \eta_0} \right)^m p_{00} \quad m \geq 0 \]  

(4.5)

implying that

\[ p_0 = (1 + \frac{\lambda_0}{\eta_0}) p_{00} \]  

(4.6)

The limit probabilities of the underlying Markov chain \( Q \), \( \pi_j = P(U(t) = j) \), satisfy

\[ \sum_{j=0}^{n} \pi_j = 1, \quad \pi_0 = \sum_{j=1}^{n} \pi_j, \text{ and } \pi_j = \pi_0 q_j \text{ for } j \geq 1. \]

Therefore \( \pi_0 = \frac{1}{2} \), and \( \pi_j = \frac{q_j}{2} \) for \( j \geq 1 \).

(Intuitively, \( \pi_0 = \frac{1}{2} \) since the MC constantly alternates between phase \( i = 0 \) and one of the other phases \( j \geq 1 \), and thus visits phase 0 half of the times). Hence, the proportion of time the system resides in phase \( i \) is given by

\[ p_i = \frac{\pi_i}{\eta_i} = \frac{q_i}{\eta_i} \quad i > 0 \]

\[ p_0 = \frac{\pi_0}{\eta_0} = \frac{1}{\eta_0} \]

(4.7)

From (4.7) it follows that

\[ \eta_i p_i = \eta_0 q_i p_0. \]  

(4.8)

Now, given \( p_0, p_{00} \) is calculated from (4.6) and all \( p_{0m}, \) for \( m \geq 0 \), are determined by (4.5).

Also,

\[ E[L_0] \equiv \sum_{m=0}^{\infty} m p_{0m} = p_{00} \sum_{m=0}^{\infty} m \left( \frac{\lambda_0}{\lambda_0 + \eta_0} \right)^m = \frac{\lambda_0}{\eta_0} \frac{1}{1 + \frac{\lambda_0}{\eta_0}} p_{00} \]

(4.9)

4.2 Generating functions

Define a set of (partial) probability generating functions (PGF):

\[ G_i(z) = \sum_{m=0}^{\infty} p_{im} z^m \quad i = 0, 1, \ldots, n \quad (0 \leq z \leq 1) \]  

(4.10)
For $i = 0$, using equations (4.1) and (4.2), we get

\[(\lambda_0(1 - z) + \eta_0)G_0(z) = \sum_{i=1}^{n} \eta_ip_i.\]  

(4.11)

Writing $G_i(1) = p_i$, and setting $z = 1$ in (4.11) we have

\[\eta_0p_0 = \sum_{i=1}^{n} \eta_ip_i.\]  

(4.12)

Equation (4.12) reflects the fact that, upon failure, in some phase $i \geq 1$, the system always moves to phase $i = 0$.

Alternatively, using (4.5),

\[G_0(z) = \sum_{m=0}^{\infty} p_{0m}z^m = p_{00} \sum_{m=0}^{\infty} \left( \frac{\lambda_0z}{\lambda_0 + \eta_0} \right)^m = p_{00} \frac{\lambda_0 + \eta_0}{\lambda_0(1 - z) + \eta_0}\]

Substituting $p_{00}(\lambda_0 + \eta_0) = \eta_0p_0$, from (4.6) and using (4.12) we have

\[G_0(z) = \eta_0p_0 = \sum_{i=1}^{n} \eta_ip_i.\]

which brings us again to (4.11).

For $i \geq 1$, using (4.3) and (4.4) we get

\[(\lambda_i + \eta_i)G_i(z) + \mu_i(G_i(z) - p_{i0}) = \lambda_i zG_i(z) + \frac{\mu_i}{z}(G_i(z) - p_{i0}) + \eta_0q_iG_0(z)\]

(4.13)

That is,

\[(\lambda_i(1 - z)z + \mu_i(z - 1) + \eta_i z)G_i(z) - \eta_0q_izG_0(z) = \mu_i(z - 1)p_{i0}\]

(4.14)

Define

\[f_0(z) = \lambda_0(1 - z) + \eta_0\]

(4.15)

\[f_i(z) = (\lambda_iz - \mu_i)(1 - z) + \eta_iz \quad i \geq 1\]

Then, the set of $n + 1$ linear equations in the $(n + 1)$ unknown PGFs $G_i(z)$ can be written as

\[A(z)g(z) = b(z)\]

(4.16)
where $A(z)$ is an $(n + 1)$-dimensional square matrix given by

$$A(z) = \begin{pmatrix} f_0(z) & 0 & 0 & \ldots & 0 \\ -\eta_0 q_1 z & f_1(z) & 0 & \ldots & 0 \\ -\eta_0 q_2 z & 0 & f_2(z) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\eta_0 q_n z & 0 & 0 & \ldots & f_n(z) \end{pmatrix}$$

and where the vectors $g(z)$ and $b(z)$ are given by

$$g(z) = \begin{pmatrix} G_0(z) \\ G_1(z) \\ \vdots \\ G_n(z) \end{pmatrix}, \quad b(z) = \begin{pmatrix} \sum_{i=1}^{n} \eta_i p_i \\ \mu_1 p_{i0} (z - 1) \\ \vdots \\ \mu_n p_{n0} (z - 1) \end{pmatrix}$$

In fact, equation (4.16) can be solved directly, using equation (4.7):

$$G_0(z) = \frac{1}{f_0(z)} \sum_{i=1}^{n} \eta_i p_i = \frac{1}{\frac{1}{\eta_0} + \sum_{k=1}^{n} \frac{q_k}{\eta_k}} f_0(z) \quad (4.17)$$

$$G_i(z) = \frac{1}{f_i(z)} \left[ \eta_0 q_i z G_0(z) + \mu_i (z - 1) p_{i0} \right], \quad i \geq 1$$

In order to complete the calculation of $G_i(z)$ we need the $n$ probabilities $p_{i0}, i = 1, 2, \ldots, n$.

Applying Cramer’s rule we can write

$$|A(z)|G_i(z) = |A_i(z)| \quad (4.18)$$

with $|A|$ as the determinant of matrix $A$, and $A_i(z)$ as the matrix given by replacing the $i$-th column of $A(z)$ with the vector $b(z)$. Clearly,

$$|A(z)| = \prod_{i=0}^{n} f_i(z) \quad (4.19)$$
$|A(z)|$ is a polynomial of degree $2n + 1$ since $f_0(z)$ is linear in $z$ and $f_i(z)$ is quadratic for every $i \geq 1$. It follows that $|A_i(z)| = 0$ whenever $|A(z)| = 0$. Using this fact we will be able to find $n$ additional relations involving the $\{p_{i0}\}$.

We now show that the polynomial $|A(z)|$ possesses exactly $n$ real roots in the open interval $(0,1)$. For $i = 0$, $f_0(z_0) = 0$ implies that $z_0 = 1 + \frac{\eta_0}{\lambda_0} > 1$. For $i \geq 1$,

\[ f_i(0) = -\mu_i < 0 \]
\[ f_i(1) = \eta_i > 0 \]
\[ f_i(\pm \infty) < 0 \]

Thus, for every $f_i(z)$, $i \geq 1$, there exists exactly one root $0 < z_i < 1$ such that $f_i(z_i) = 0$. Therefore there are exactly $n$ distinct roots in $(0,1)$ $z_1, z_2, \ldots, z_n$, for $|A(z)| = 0$. The root $z_i \in (0,1)$ is derived from $f_i(z_i) = 0$ and it is given by

\[ z_i = \frac{(\lambda_i + \mu_i + \eta_i) - \sqrt{(\lambda_i + \mu_i + \eta_i)^2 - 4\lambda_i\mu_i}}{2\lambda_i} \]

Considering $z_j$ as one of those roots of $|A(z)|$, equation (4.18) yields

\[ |A_i(z_j)| = 0; \quad j = 1, 2, \ldots, n; \quad i = 1, 2, \ldots, n \quad (4.20) \]

From (4.18) it follows that, for every $z_j$ and $1 \leq i, k \leq n$

\[ \frac{|A_i(z_j)|}{|A_k(z_j)|} = \frac{G_i(z_j)}{G_k(z_j)} = \text{Constant} \]

Hence, equation (4.20) gives only one independent equation for each $z_j$. Thus, the $n$ equations for $p_{i0}$, $i = 1, 2, \ldots, n$ are obtained from

\[ |A_i(z_i)| = 0 \quad i = 1, 2, \ldots, n \]

Using (4.12), the polynomial $|A_i(z_i)|$ is given by

\[ |A_i(z_i)| = \prod_{k=1;k \neq i}^{n} f_k(z_i)[\mu_ip_{i0}(z_i - 1)f_0(z_i) + \eta_0^2p_{i0}g_i z_i] \quad (4.21) \]

Therefore, $|A_i(z_i)| = 0$ yields

\[ p_{i0} = \frac{\eta_0^2p_{i0}g_i z_i}{|\lambda_0(1 - z_i) + \eta_0|\mu_i(1 - z_i)} \quad i \geq 1 \quad (4.22) \]
4.3 Mean queue size

Let \( G'_i(1) = E[L_i] = \sum_{m=1}^{\infty} m p_{im}, \ i = 0, 1, \ldots, n. \) Then, taking derivatives of (4.11) and (4.13) at \( z = 1, \) we obtain (using (4.6) and (4.7))

\[
E[L_0] = \frac{\lambda_0}{(1 + \eta_0^2)(\eta_0 + \sum_{k=1}^{n} \eta_k)} = \frac{\lambda_0}{\eta_0} (1 + \frac{\lambda_0}{\eta_0}) p_{00} \quad (4.23)
\]

\[
(-\lambda_i + \mu_i + \eta_i) p_i + \eta_i E[L_i] - \eta_0 q_i (p_0 + E[L_0]) = \mu_i p_i \quad i \geq 1 \quad (4.24)
\]

This leads to

\[
E[L_i] = \frac{1}{\eta_i} \left[ \mu_i p_{i0} + \left( \frac{\lambda_0}{\eta_0} + \frac{\lambda_i}{\eta_i} - \frac{\mu_i}{\eta_i} \right) \frac{q_i}{1 + \sum_{k=1}^{n} \eta_k} \right]
\]

\[= \frac{1}{\eta_i} [\mu_i p_{i0} + \left( \frac{\lambda_0}{\eta_0} + \frac{\lambda_i}{\eta_i} - \frac{\mu_i}{\eta_i} \right) p_i \eta_i] \quad i \geq 1 \quad (4.25)
\]

The total number of customers in the system is

\[
E[L] = \sum_{i=0}^{n} E[L_i]
\]

\[= \frac{\lambda_0}{\eta_0} (1 + \frac{\lambda_0}{\eta_0}) p_{00} + \sum_{i=1}^{n} \frac{1}{\eta_i} \left[ \frac{\eta_0^2 p_{00} \eta_i z_i}{(\lambda_0(1 - z_i) + \eta_0)(1 - z_i)} + \frac{\lambda_0}{\eta_0} + \frac{\lambda_i}{\eta_i} - \frac{\mu_i}{\eta_i} \right] \quad (4.26)
\]

4.4 Number of customers cleared

Consider the following performance measure: let \( R \) be the number of customers cleared from the system per unit time. Then

\[
E[R] = \sum_{i=1}^{n} \eta_i \sum_{m=1}^{\infty} m p_{im} = \sum_{i=1}^{n} \eta_i E[L_i]
\]

The fraction of customers receiving full service is therefore

\[
\frac{\lambda - E[R]}{\lambda} = 1 - \frac{E[R]}{\lambda}
\]

4.5 Sojourn Times

Let \( W_{im} \) be the sojourn time of a customer that arrives to the system when it is in state \((i, m)\). In a FCFS regime, future arrivals do not affect the sojourn time of a waiting customer.
Hence,
\[ E[W_{im}] = \frac{\mu_i}{\mu_i + \eta_i} \left( \frac{1}{\mu_i + \eta_i} + E[W_{i,m-1}] \right) + \frac{\eta_i}{\mu_i + \eta_i} \left( \frac{1}{\mu_i + \eta_i} + 0 \right) \]
where \( W_{im} = 0 \) for \( m < 0 \). Thus,
\[ E[W_{im}] = \frac{1}{\mu_i + \eta_i} + \frac{\mu_i}{\mu_i + \eta_i} E[W_{i,m-1}] \quad m \geq 1 \quad (4.27) \]
This leads to
\[ E[W_{im}] = \frac{1}{\eta_i} \left[ 1 - \left( \frac{\mu_i}{\mu_i + \eta_i} \right)^{m+1} \right] \quad (4.28) \]
Indeed, when in state \((i, m)\), the time until departure, whether as a result of a failure, or as a result of service completion, is the minimum of two variables: (a) time to failure, distributed exponentially with parameter \( \eta_i \), and (b) sum of \((m + 1)\) service completions, which is Erlang\((\mu_i, m + 1)\).

Thus, let
\[ X_i \sim \text{Erlang}(\mu_i, m + 1) \]
\[ Y_i \sim \text{Exp}(\eta_i) \]
so that
\[ W_{im} = \text{min}(X_i, Y_i) \]
Then
\[ P(W_{im} \geq w) = P(X_i \geq w)P(Y_i \geq w) = \sum_{k=0}^{m} e^{-\mu_i w} \frac{(\mu_i w)^k}{k!} \]
Therefore,
\[ E[W_{im}] = \int_{0}^{\infty} P(W_{im} \geq w)dw = \int_{0}^{\infty} \sum_{k=0}^{m} e^{-\mu_i w} \frac{(\mu_i w)^k}{k!} e^{-\eta_i w} dw \]
\[ = \sum_{k=0}^{m} \frac{\mu_i^k}{(\mu_i + \eta_i)^{k+1}} \int_{0}^{\infty} e^{-(\mu_i + \eta_i) w} \frac{(\mu_i + \eta_i)^{k+1} w^k}{k!} dw = \frac{1}{\eta_i} \left[ 1 - \left( \frac{\mu_i}{\mu_i + \eta_i} \right)^{m+1} \right] \]
Define \( E[W] \) as the mean sojourn time of an arbitrary customer. Then,
\[ E[W] = \sum_{i=0}^{n} \sum_{m=0}^{\infty} p_{im} E[W_{im}] \]
In fact, using Little’s law, \( E[W] = \frac{1}{\lambda} E[L] \) where \( \hat{\lambda} = \sum_{i=0}^{n} \lambda_i p_i \).
4.6 A special case: two phases with identical arrival rate

A special case of the general model introduced in section 4.1 is when \( n = 1 \). In this case there is only one active phase: an \( M/M/1 \) queue; and one ’failure’ phase. The arrival rates are \( \lambda_0 = \lambda_1 = \lambda \). Also, since \( n = 1 \), \( q_1 = 1 \). This model is a special case of Yechiali [16] where an \( M/M/1 \) system with breakdowns and cutomers’ impatience is studied. Assuming in [16] that \( \xi = 0 \), the two models coincide.

The steady-state balance equations are:

\[
\begin{align*}
    i = 0, & \quad m = 0 \quad (\lambda + \eta_0) p_{00} = \eta_1 p_{1} \\
    m \geq 1 & \quad (\lambda + \eta_0) p_{0m} = \lambda p_{0,m-1} \\
    i = 1, & \quad m = 0 \quad (\lambda + \eta_1) p_{10} = \mu_1 p_{11} + \eta_0 p_{00} \\
    m \geq 1 & \quad (\lambda + \mu_1 + \eta_1) p_{1m} = \lambda p_{1,m-1} + \mu_1 p_{1,m+1} + \eta_0 p_{0m}
\end{align*}
\]

The limit probabilities in (4.7) are translated to

\[
\begin{align*}
    p_{0*} &= \frac{\eta_1}{\eta_0 + \eta_1} \\
    p_{1*} &= \frac{\eta_0}{\eta_0 + \eta_1}
\end{align*}
\]  

(4.29)

The generating functions given in (4.11) and (4.14) are:

\[
\begin{align*}
    (\lambda(1 - z) + \eta_0) G_0(z) &= \eta_1 p_{1} \\
    (\lambda(1 - z) + \mu_1(z - 1) + \eta_1 z) G_1(z) - \eta_0 z G_0(z) &= \mu_1(z - 1) p_{10}
\end{align*}
\]  

(4.30)

From equations (4.6) and (4.30) we get that

\[
    p_{00} = \frac{\eta_1}{\eta_0 + \eta_1} \cdot \frac{\eta_0}{\eta_0 + \lambda}
\]  

(4.31)

Using the generating functions, and equation (4.22)

\[
p_{10} = \frac{p_{00}^2 z_1}{(\lambda(1 - z_1) + \eta_0) \mu_1(1 - z_1)}
\]  

(4.32)

where \( z_1 \) is the root of \( f_1(z) \), as discussed in the previous section.
Differentiating (4.30) at \( z = 1 \) we get:

\[
E[L_0] = \frac{\lambda p_0}{\eta_0}
\] (4.33)

and

\[
(-\lambda + \mu_1 + \eta_1)p_1 + \eta_1 E[L_1] - \eta_0 (p_0 + E[L_0]) = \mu_1 p_{10}
\] (4.34)

Equation (4.33) equates the mean rate of arrivals \( \lambda p_0 \) to phase 0 to the mean rate of departures \( \eta_0 E[L_0] \) from this phase. Rewriting (4.34) leads to

\[
E[L_1] = \frac{1}{\eta_1}[(\lambda - \mu_1 - \eta_1)p_1 + \eta_0 (p_0 + \frac{\lambda p_0}{\eta_0}) + \mu_1 p_{10}]
\] (4.35)

Using equations (4.12) and (4.33), equation (4.35) can be written as

\[
\eta_1 E[L_1] = \lambda p_1 + \eta_0 E[L_0] - \mu_1 (p_1 - p_{10})
\] (4.36)

Again, equation (4.36) equates the rates of inflow and outflow of phase 1. In fact, \( \eta_1 E[L_1] \) is the expected number of customers cleared from the system per unit time (see Section 4.4).

### 4.7 Arrival stops when the system is down

Another special case of the general model of section 4.1 is when the arrival process stops whenever the system is down. That is, \( \lambda_0 = 0 \), implying that \( p_{0m} = 0, \forall m > 0 \). Using the steady-state balance equation (4.1) of the general model we get, for \( i = 0 \),

\[
\eta_0 p_{00} = \sum_{i=1}^{n} \eta_i p_i
\] (4.37)

Equation (4.3) remains unchanged, but for \( i \geq 1 \) and \( m \geq 1 \), (4.4) is replaced by

\[
(\lambda_i + \mu_i + \eta_i)p_{im} = \lambda_i p_{i,m-1} + \mu_i p_{i,m+1}
\] (4.38)

The set of probabilities \( p_i \) is given by (4.7).

From (4.11) and (4.14) the set of PGFs is given by, for \( i = 0 \),

\[
\eta_0 G_0(z) = \sum_{i=1}^{n} \eta_i p_i
\] (4.39)

30
and, for $i \geq 1$,
\[
(\lambda_i (1-z)z + \mu_i (z-1) + \eta_i z) G_i(z) = \mu_i (z-1)p_{i0} + \eta_0 p_{00} z \quad (4.40)
\]

Define
\[
f_0(z) = \eta_0
f_i(z) = (\lambda_i z - \mu_i)(1-z) + \eta_i z; \quad i = 1, \ldots, n
\]

Then, similarly to (4.16)
\[
A(z)g(z) = b(z)
\]

where $A(z)$ is given by
\[
A(z) = \begin{pmatrix}
f_0(z) & 0 & 0 & \ldots & 0 \\
0 & f_1(z) & 0 & \ldots & 0 \\
0 & 0 & f_2(z) & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & f_n(z)
\end{pmatrix}
\]

and the vectors $g(z)$ and $b(z)$ are given by
\[
g(z) = \begin{pmatrix}
G_0(z) \\
G_1(z) \\
\vdots \\
G_n(z)
\end{pmatrix}
b(z) = \begin{pmatrix}
\eta_0 p_{00} \\
\mu_1 p_{10} (z-1) + \eta_0 q_1 p_{00} z \\
\vdots \\
\mu_n p_{n0} (z-1) + \eta_0 q_n p_{00} z
\end{pmatrix}
\]

The equivalent of (4.17) is
\[
G_0(z) = p_{00}
G_i(z) = \frac{1}{f_i(z)} [\mu_i (z-1)p_{i0} + \eta_0 q_i p_{00} z] \quad i \geq 1 \quad (4.41)
\]
Differentiating (4.40) leads to

\[ E[L_0] = 0 \]  
\[ \eta_i E[L_i] = \mu_i p_{io} + \eta_0 q_ip_{00} + (\lambda_i - \mu_i - \eta_i)p_0. \quad i \geq 1 \]  

(4.42)

Since in this case \( p_{0i} = p_{00} \), we have that (see 4.8) \( \eta_i p_0 = \eta_0 q_ip_{00} \). Substituting this relation in (4.42) yields

\[ \eta_i E[L_i] + \mu_i (p_i - p_{0i}) = \lambda_i p_i. \]  

(4.43)

which, again equates the inflow and outflow rates of phase \( i \), only this time it does not concerning the phase transition intensity.

### 4.8 Numerical results

We present a few numerical examples of the second model, as represented in this section.

We set the number of phases to be \( n = 4 \): there are 4 ‘regular’ phases and a failure phase.

(i) \( \lambda_i = 0.4 \) for all \( i \)

<table>
<thead>
<tr>
<th>phase</th>
<th>( \lambda_i )</th>
<th>( \mu_i )</th>
<th>( \eta_i )</th>
<th>( q_i )</th>
<th>( \pi_i )</th>
<th>( p_0 )</th>
<th>( z )</th>
<th>( p_{0i} )</th>
<th>( E[L_i] )</th>
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<td>0.0266</td>
<td>11.1732</td>
<td></td>
</tr>
<tr>
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<td>0.4</td>
<td>0.3</td>
<td>0.04</td>
<td>0.3</td>
<td>0.0838</td>
<td>0.6000</td>
<td>0.0019</td>
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<td></td>
</tr>
<tr>
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</tr>
<tr>
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<td>0.0023</td>
<td>0.3857</td>
<td></td>
</tr>
<tr>
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</table>

\[ E[L] = 18.1637 \]
(ii) \( \mu_i = 0.5 \) for all \( i \)

<table>
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<tr>
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<th>( \lambda_i )</th>
<th>( \mu_i )</th>
<th>( \eta_i )</th>
<th>( q_i )</th>
<th>( \pi_i )</th>
<th>( p_i )</th>
<th>( z )</th>
<th>( p_{i0} )</th>
<th>( E[L_i] )</th>
</tr>
</thead>
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</tbody>
</table>

\[ E[L] = 21.435 \]

(iii) \( \eta_i = 0.05 \) for all \( i \)

<table>
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<th>phase</th>
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<th>( \mu_i )</th>
<th>( \eta_i )</th>
<th>( q_i )</th>
<th>( \pi_i )</th>
<th>( p_i )</th>
<th>( z )</th>
<th>( p_{i0} )</th>
<th>( E[L_i] )</th>
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<tbody>
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<td>0</td>
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<td>0.15</td>
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<td>0.20</td>
<td>0.20</td>
<td>0.2605</td>
<td>0.0021</td>
<td>4.0084</td>
</tr>
</tbody>
</table>

\[ E[L] = 12.428 \]

Again it is clear that in this case \( \pi_i = p_i \).
(iv) The case where $\lambda_0 = 0$

<table>
<thead>
<tr>
<th>phase</th>
<th>$\lambda_i$</th>
<th>$\mu_i$</th>
<th>$\eta_i$</th>
<th>$q_i$</th>
<th>$\pi_i$</th>
<th>$p_i$</th>
<th>$z$</th>
<th>$p_{00}$</th>
<th>$E[L_i]$</th>
</tr>
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<tbody>
<tr>
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<td>0</td>
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<td>−</td>
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<td>0</td>
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$E[L] = 6.1931$

(v) A case where $\lambda_i/\mu_i = c$ for all $i$

<table>
<thead>
<tr>
<th>phase</th>
<th>$\lambda_i$</th>
<th>$\eta_i$</th>
<th>$q_i$</th>
<th>$\pi_i$</th>
<th>$p_i$</th>
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</thead>
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<td>0.05</td>
<td>0.4</td>
<td>0.20</td>
<td>0.1132</td>
</tr>
</tbody>
</table>

Clearly, $\{\pi_i\}$ and $\{p_i\}$ are independent of $\{\lambda_i\}$ and $\{\mu_i\}$

Figure 4 below depicts the increase of the various $E[L_i]$ with $c$. Note that the total mean queue size is finite even when $c$ approaches 1. This follows because, upon failure, all customers are cleared from the system.
Figure 4: Mean queue sizes in the $M/M/1$ model, when $\lambda_i/\mu_i = c$
5 Transportation implementation

5.1 $M/M/\infty$ queue model

Consider the $M/M/\infty$ queue in random environment as a model for a transportation roadway. The time required by a vehicle to pass the road section is taken as its service time and assumed to be exponentially distributed with mean $1/\mu$. Since a road section can have many vehicles using it at a given time, it can be approximately modeled as an $M/M/\infty$ queue. Each phase in our $M/M/\infty$-type queue model represents a different condition of the road. During an incident, the road conditions deteriorate such that the service rates decrease and the arrival rate may also change. As soon as the incident is cleared and the road section is restored to its normal condition and the incident is cleared, the arrival and service rates return to their normal level. The changes in road conditions, which influence the inflow and outflow rates are represented by transitions between phases in the model. Using these analogies one can refer to the $n$-phase random environment $M/M/\infty$ queue model, described in section 3, as representing the stochastic behavior of the road traffic.

Indeed, often there is a direct connection between the road phase and the arrival and service rates. For example, in a good road condition, more vehicles are attracted to use the road, causing a higher value of $\lambda_i$, or vice versa. Thus, one can consider the case where $\lambda_i/\mu_i = c$ for all $i$, although the absolute values of $\lambda_i$ and $\mu_i$ can be high or low. Hence, using Theorem 1, we conclude that $p_{im}$ follows the Poisson distribution. The system possesses properties of a standard $M/M/\infty$ queue and, using Corollary 5, we find that the mean number of vehicles on a road is given by $E[L] = \lambda_i/\mu_i = c$.

We present numerical examples related to the model. We consider the number of phases to be $n = 2$. Let $d$ be the length of the road section, and let $v$ be the average speed of each vehicle on this section. The total time it takes a vehicle to pass the road section is $\frac{1}{\mu} = \frac{d}{v}$. Let phase $i = 2$ represent the standard road section with no incidents, and having a standard speed $v$. Let phase $i = 1$ be the bad condition phase, representing the road section in a case of an incident. Changing the transition parameter from phase 2 to phase 1, i.e. $\eta_2$, and
using arbitrary values for the various parameters presented below, we get the graph in figure 5. Note that, when calculating $E[L]$ we do not take into consideration the safety distance, required between two moving vehicles. In reality this distance approximately increases as a quadratic function. We further discuss this issue in the Appendix.

<table>
<thead>
<tr>
<th>phase</th>
<th>$\lambda_i$</th>
<th>$q_{i1}$</th>
<th>$q_{i2}$</th>
<th>$\pi_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4</td>
<td>1</td>
<td>0</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>0.4</td>
<td>0</td>
<td>1</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Each line in the graph is a calculation of the mean queue size of phase 2, with a change of the standard speed $v$. The graph shows that the larger the speed of the vehicle over the road section, the smaller the queue size is. It shows the relation between the 'repair' rate in which the incident is taking care of and the mean queue size.

![Figure 5: Mean queue size as a function of the transition rate $\eta_2$ in the $M/M/\infty$ model, when $\mu_2$ changes according to the speed $v$.](image)

Figure 5: Mean queue size as a function of the transition rate $\eta_2$ in the $M/M/\infty$ model, when $\mu_2$ changes according to the speed $v$. 

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5.2 \( M/M/1 \) queue model

A different approach is to consider the \( M/M/1 \) queue in random environment as representing a road section. Since there are many potential vehicles to use a given road, and a small probability of each one to use this road at a given time interval, the arrival process may be approximated by the Poisson distribution. In this model there is only a single server, e.g. a traffic light. When there is an incident such as an accident, the vehicles are directed to another roadway. Therefore the situation can be considered as clearing the customers from the queue. New vehicles joining after the occurance of the incident, will wait until the incident is taken care of, and the road is again ready for use.

Since there is no direct transition from one phase to another, like in the \( M/M/\infty \) model, the only time that the rates of arrival and service can be changed is following an incident. The number of vehicles on the road changes during the incident, and this affects the arrival and service rates. It was shown in equation (4.22) that the probability of serving all vehicles, represented indirectly by the value of \( p_{i0} \), depends heavily on the ability to fix the incident as quickly as possible (high \( \eta_0 \)). Note also that the sojourn time of a customer when the system is in phase \( i \), calculated in (4.28), depends only on the phase-dependent service rate and the duration of time the system resides in phase \( i \).

We present a numerical example where we consider two phases: a failure phase represented by phase \( i = 0 \), and a standard phase \( i = 1 \). Following the numerical example in section 5.1, using the same arbitrary values, and having \( v \) as the speed while driving along the road section, we get the graph in figure 6.
Figure 6: Mean queue size as a function of $\eta_2$ in the M/M/1 model, when $\mu_2$ changes according to the speed $v$.

Each line in Figure 6 represents a calculation of the mean queue size of phase $i = 2$ with a different speed $v$. The graph demonstrates the affect of the transition rate, $\eta_2$, on the mean queue size for various values of the vehicle’s speed. In this model the queue size is almost un-effected by the speed values, since in any case of an incident the customers are cleared and evacuated from the queue.
In this research, we propose two kinds of queueing models describing the traffic flow on a roadway. Each model includes queue phases, where the various phases represent different situations of the traffic condition and flow. We study cases where every phase is described either by an $M/M/\infty$ queue or by an $M/M/1$ queue, and explore the corresponding solutions. For a special case where $\lambda_i/\mu_i = c$ for every phase $i$, we reveal a full compatibility between our $n$-phase $M/M/\infty$ model and the regular $M/M/\infty$ queue. Our work proposes a general approach for cases with $n \geq 2$ phases. Considering road traffic, we explicitly reveal the dependence between the transition rates between phases and the number of vehicles in the system. Accordingly, considering safety requirements of distance between vehicles on the road depending on their speed, one can find the optimal driving speed for a road section, considering incidents which might happen while the vehicles are driving along a given road section.
References


A Appendices

A.1 Numerical calculation: the number of vehicles passing a road section

We find a relation between the number of vehicles on a road section and vehicle’s speed. Let \( v \) be the speed of a vehicle on a given road section of length \( d \). Assume all vehicles drive in the same speed. Let \( l \) be the length of a vehicle and \( y(v) \) the required safety distance between two vehicles that move at speed \( v \). Then the maximum number of vehicles traveling at speed \( v \) in a road section of length \( d \) is

\[
n = \frac{d}{l + y(v)}
\]

The safety distance is the stopping distance, the minimum distance that a vehicle can be brought to rest in an emergency from the moment that the driver notices danger ahead. The stopping distance is calculated as:

\[
\text{stopping distance} = \text{thinking distance} + \text{braking distance}
\]

The thinking distance is related to the reaction time, the time taken for a driver to react, which is about 0.1-0.3 seconds. As a general rule, the braking distance becomes four times greater as the speed of the car is doubled (see [13]). In real situations, the actual stopping distance depends also on several other factors such as the condition of the road, the vehicle’s mechanical condition, and physical and mental states of the driver.

Assuming deceleration value of \( a = 9 \text{ m/sec}^2 \), the braking distance, \( b \), may be approximately calculated as \( b = \frac{v^2}{2a} = \frac{v^2}{18} \). Thus, at \( v = 14 \text{ m/sec} \), \( b = 10.9 \text{ m} \). Assuming safety distance as \( b/2 \), we get that at driving speed \( v = 50.4 \text{ km/h} \), the number of vehicles traveling on a 1 km road is

\[
n = \frac{1000}{l + \frac{b}{2}} = \frac{1000}{l + \frac{v^2}{2a} \cdot \frac{1}{2}} = 106
\]
Hence, having $C$ as a converting factor between units, the number of vehicles driving along a road section in 1 hour is

$$L(v) = v \cdot n = C \cdot \frac{v}{l + \frac{v^2}{4a}} = 5342$$

![Figure 7: The number of vehicles as a function of the vehicle's speed.](image)

Figure 7 represents the dependence of the number of vehicles driving along a road section during 1 hour on the vehicles’ speed. It is also shown that there is an optimal speed $v^*$ with the greatest number of vehicles. Differentiating $L(v)$, we find that $v^* = 2\sqrt{al} = 12$ m/sec = 47.132 km/h. This is the optimal speed for having the greatest number of vehicles driving along a road section, one by one, with a safety distance of $b/2$ between every two vehicles.